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Parity Decision Tree Complexity and 4-Party Communication Complexity of XOR-functions Are Polynomially Equivalent

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Abstract: In this note, we study the relationship between the *parity decision tree complexity* of a boolean function f, denoted by $\mathrm{D}_{\oplus}(f)$, and the *k-party number-in-hand multiparty communication complexity* of the XOR-functions $F_k(x_1,\ldots,x_k) \stackrel{\text{def}}{=} f(x_1 \oplus \cdots \oplus x_k)$, denoted by $\mathrm{CC}^{(k)}(F_k)$. It is known that $\mathrm{CC}^{(k)}(F_k) \leq k \cdot \mathrm{D}_{\oplus}(f)$ because the players can simulate the parity decision tree that computes f. In this note, we show that

$$\mathrm{D}_{\oplus}(f) = \mathcal{O}\left(\mathrm{CC}^{(4)}(F_k)^5\right).$$

Our main tool is a recent result from additive combinatorics due to Sanders [14]. As $\mathrm{CC}^{(k)}(F_k)$ is non-decreasing as k grows, the parity decision tree complexity of f and the communication complexity of the corresponding k-argument XOR-functions are polynomially equivalent whenever $k \geq 4$.

Remark: After a first version of this paper was finished, we were informed that Hatami and Lovett had already discovered the same result a few years ago, without writing it up.

1 Introduction

Communication complexity and the Log-Rank Conjecture for XOR-functions Communication complexity quantifies the minimum amount of communication needed for computation when

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inputs are distributed among different parties [18, 5]. In the model of two-party communication, Alice and Bob hold inputs x and y, respectively, and they are supposed to compute the value of a function $F(x,y)$ using as little communication as possible. One of the central problems in communication complexity is the *Log-Rank Conjecture*. This conjecture, proposed by Lovász and Saks in [8], asserts that the communication complexity of F and $\log \operatorname{rank}(M_F)$ are polynomially equivalent for any 2-argument total boolean function F, where $M_F = [F(x,y)]_{x,y}$ is the communication matrix of F. Readers may refer to [17] for more discussion on the conjecture. The conjecture is notoriously hard to attack. It was shown in [11] 30 years ago that $\log \operatorname{rank}(M_F)$ is a lower bound on the deterministic communication complexity of F. The state of the art is

$$\mathrm{CC}^{(2)}(F) = \mathcal{O}\left(\sqrt{\operatorname{rank}(M_F)} \log \operatorname{rank}(M_F)\right),$$

where $\mathrm{CC}^{(2)}(F)$ stands for the two-party deterministic communication complexity of F. It is from a recent breakthrough due to Lovett [10]. The largest known gap between $\mathrm{CC}^{(2)}(F)$ and $\log \operatorname{rank}(M_F)$ is $\mathrm{CC}^{(2)}(F) \geq \tilde{\Omega}\left(\log \operatorname{rank}(M_F)^2\right)$ due to Göös, Pitassi and Watson [4].

In [21], Zhang and Shi initiated the study of the Log-Rank Conjecture for a special class of functions called *XOR-functions*.

Definition 1.1. We say a k-argument function $F_k : (\{0,1\}^n)^k \to \{0,1\}$ is an XOR-function if there exists a function $f : \{0,1\}^n \to \{0,1\}$ such that $F_k = f \circ \oplus^k$. Namely, $F_k(x_1,\ldots,x_k) = f(x_1 \oplus \ldots \oplus x_k)$ for any $x_1,\ldots,x_k \in \{0,1\}^n$, where $\oplus$ is the bitwise xor.

XOR-functions include many important examples, such as Equality and Hamming distance. The communication complexity of XOR functions has been studied extensively in the last decade [20, 6, 12, 7, 17, 19]. A nice feature of XOR-functions is that the rank of the communication matrix M_F is exactly the *Fourier sparsity* of f.

Fact 1.2. *[2] For any boolean function f and the associated XOR-function F_2 given in Definition 1.1, it holds that $\operatorname{rank}(M_{F_2}) = \|\hat{f}\|_0$, where $\|\hat{f}\|_0$ is the Fourier sparsity of f (see Section 2 for the definition).*

Therefore, the Log-Rank Conjecture for XOR-functions is equivalent to the question whether there exists a protocol computing F with communication $\log^{\mathcal{O}(1)} \|\hat{f}\|_0$. However, the Log-Rank Conjecture is still difficult for this special class of functions. One nice approach proposed in [20] is to design a parity decision tree (PDT) to compute f. PDTs allow to query the parity of any subset of input variables. For any k-argument XOR-function F given in Definition 1.1, we can construct a communication protocol by simulating the PDT for f, with communication k times the PDT complexity of f. It is therefore sufficient to show that $\mathrm{D}_{\oplus}(f) \leq \log^{\mathcal{O}(1)} \|\hat{f}\|_0$. Using such an approach, the Log-Rank Conjecture has been established for several subclasses of XOR-functions [12, 17].

One question regarding this approach is whether $\mathrm{D}_{\oplus}(f)$ and $\mathrm{CC}^{(2)}(F)$ are polynomially equivalent. Is it possible to design a protocol for F much more efficient than simulating the parity decision tree of f?

Conjecture 1.3. *For any boolean function $f : \{0,1\}^n \to \{0,1\}$ and the associated XOR-function F_2, there is a constant c such that $\mathrm{CC}^{(2)}(F_2) = \mathcal{O}(\mathrm{D}_{\oplus}(f)^c)$*

If this holds, then the Log-Rank Conjecture for XOR-functions is equivalent to a question regarding parity decision trees. Namely, whether $\mathrm{D}_{\oplus}(f) = \mathcal{O}\left(\log\left(\|\hat{f}\|_0\right)^c\right)$ for some constant c. In this note, we prove a weaker variant of the above conjecture. Given a total boolean function f, we may also consider the communication complexity of the associated k-argument XOR-function F_k given in Definition 1.1 in the model of *number-in-hand k-party communication*, which is denoted by $\mathrm{CC}^{(k)}(F_k)$. It is easy to see that $\mathrm{CC}^{(2)}(F_2) \leq \mathrm{CC}^{(3)}(F_3) \leq \ldots$ and $\mathrm{CC}^{(k)}(F_k) \leq k \cdot \mathrm{D}_{\oplus}(f)$. Our main result in this note is that $\mathrm{CC}^{(k)}(F_k)$ and $\mathrm{D}_{\oplus}(f)$ are polynomially equivalent whenever $k \geq 4$.

Theorem 1.4. *For any boolean function $f : \{0,1\}^n \to \{0,1\}$, let F_4 be the associated 4-argument XOR-function given in Definition 1.1. It holds that*

$$\mathrm{D}_{\oplus}(f) = \mathcal{O}\left(\mathrm{CC}^{(4)}(F_4)^5\right).$$

Our Techniques

To show the main theorem, it suffices to construct an efficient PDT for f if the communication complexity of F is small. We adapt a protocol introduced by Tsang et al. [17]. The main step is to exhibit a large monochromatic affine subspace for f if the communication complexity of F is small. To this end, we adapt the quasipolynomial Bogolyubov-Ruzsa lemma [14], which says that $4A \stackrel{\text{def}}{=} A + A + A + A$ contains a large subspace if $A \subseteq \mathbb{F}_2^n$ is large.

Related Work

A large body of work has been devoted to the Log-Rank Conjecture for XOR-functions. After almost a decade of efforts, the conjecture has been established for several classes of XOR-function, such as symmetric functions [20], monotone functions and linear threshold functions [12], constant $\mathbb{F}_2$-degree functions [17].

A different line of work close to ours is the *simulation theorem* in [13, 20, 15, 3, 4]. They study the relationship between the (regular) decision tree complexity of function f and the communication complexity of $f \circ g^n$ where g is a 2-argument function of small size. The simulation theorem asserts that the optimal protocol for $f \circ g^n$ is to simulate the decision tree that computes f if g is a hard function. Simulation theorems have been established in various cases, when g is bitwise AND or OR [15], Inner-Product [3], Index Function [13, 4]. Our work gives a new simulation theorem when g is an XOR function.

After this work was put online, the author was informed that Hatami and Lovett discovered Theorem 1.4 (using the same idea) a couple years ago without writing it up. Since our work is independent to theirs, we believe it is worth having a complete proof of the main theorem.

2 Preliminaries

All logarithms in this note are base 2. Given $x, y \in \{0,1\}^n$, we define the inner product $x \cdot y \stackrel{\text{def}}{=} \sum_{i=1}^n x_i y_i \mod 2$. For simplicity, we write $x + y$ for $x \oplus y$.

Complexity measures. Given a boolean function $f : \{0,1\}^n \to \{0,1\}^n$, it can be viewed as a polynomial in $\mathbb{F}_2$, and $\deg_2(f)$ is used to denote its $\mathbb{F}_2$-degree.

Definition 2.1. Given a function $f : V \to \mathbb{F}_2$, where V is an affine subspace of $\mathbb{F}_2^n$, the parity certificate complexity of f on x is defined to be

$$\mathrm{C}_{\oplus}(f, x) \stackrel{\text{def}}{=} \min\{\mathrm{codim}(H) \ : \ H \subseteq V \text{ is an affine subspace where } f \text{ is constant and } x \in H\}$$

where $\operatorname{codim}(H) \stackrel{\text{def}}{=} \dim V - \dim H$. The minimum parity certificate complexity is defined as $\mathrm{C}_{\oplus,\min}(f) \stackrel{\text{def}}{=} \min_x \mathrm{C}_{\oplus}(f,x)$.

Definition 2.2. Given a boolean function $f:\{0,1\}^n \to \{0,1\}$. We view it as a polynomial in $\mathbb{F}_2$. The *linear rank* of f, denoted $\mathrm{rk}(f)$, is the minimum integer r, such that f can be expressed as $f = \sum_{i=1}^{r} l_i f_i + f_0$, where $\deg_2(l_i) = 1$ for $1 \le i \le r$ and $\deg_2(f_i) < \deg_2(f)$ for $0 \le i \le r$.

Definition 2.3. A parity decision tree (PDT) for a boolean function $f:\{0,1\}^n \to \{0,1\}$ is a tree with internal nodes associated with a subset $S \subseteq [n]$ and each leaf associated with an answer in $\{0,1\}$. To use a parity decision tree to compute f, we start from the root and follow a path down to a leaf. At each internal node, we query the parity of the bits with the indices in the associated set and follow the branch according to the answer to the query. Output the associated answer when we reach the leaf. The deterministic parity decision tree complexity of f, denoted by $\mathrm{D}_{\oplus}(f)$, is the minimum number of queries needed on a worst-case input by a PDT that computes f correctly.

Definition 2.4. In the model of number-in-hand multiparty communication, there are k players $\{P_1,\ldots,P_k\}$ and a k-argument function $F:(\{0,1\}^n)^k \to \{0,1\}$. Player P_i is given an n-bit input $x_i \in \{0,1\}^n$ for each $i \in [k]$. The communication is in the blackboard model. Namely, every message sent by a player is written on a blackboard visible to all players. The communication complexity of f in this model, denoted by $\mathrm{CC}^{(k)}(F)$, is the least number of bits needed to be communicated to compute f correctly.

One way to design a protocol for the k-argument XOR-function $F_k = f \circ \oplus^k$ is to simulate a parity decision tree that computes f.

Fact 2.5. *Let $f:\{0,1\}^n \to \{0,1\}$ be a boolean function and $F_k = f \circ \oplus^k$ given in Definition 1.1. It holds that $\mathrm{CC}^{(k)}(F) \le k \cdot \mathrm{D}_{\oplus}(f)$.*

Fourier analysis. For any real-valued function $f:\{0,1\}^n \to \mathbb{R}$, the Fourier coefficients are defined as $\hat{f}(s) \stackrel{\text{def}}{=} \frac{1}{2^n}\sum_x f(x)\chi_s(x)$ for $s \in \{0,1\}^n$, where $\chi_s(x) \stackrel{\text{def}}{=} (-1)^{s\cdot x}$. The function f can be decomposed as $f = \sum_s \hat{f}(s)\chi_s$.The Fourier sparsity $\|\hat{f}\|_0$ is the number of nonzero Fourier coefficients of f.

Fact 2.6. *For all $f:\{0,1\}^n \to \{0,1\}$, it holds that $\deg_2(f) \le \log\|\hat{f}\|_0$.*

Let $V \subseteq \mathbb{F}_2^n$ be an affine subspace and $f: V \to \mathbb{F}_2$ be a boolean function. A complexity measure $m(f)$ of f is *downward non-increasing* if $m(f') \le m(f)$ for any subfunction f' obtained by restricting f to an affine subspace of V. For instance, $\deg_2(\cdot)$ is downward non-increasing.

Fact 2.7. *[17] If $\mathrm{rk}(\cdot) \le \mathrm{m}(\cdot)$ for some downward non-increasing complexity measure m, then it holds that $\mathrm{D}_{\oplus}(f) \le m(f)\cdot\log\deg_2(f)$. Combining with Fact 2.6, we have $\mathrm{D}_{\oplus}(f) \le m(f)\cdot\log\|\hat{f}\|_0$.*

Fact 2.8. *[17] For all non-constant $f:\mathbb{F}_2^n \to \mathbb{F}_2$, it holds that $\mathrm{rk}(f) \le \mathrm{C}_{\oplus,\min}(f)$.*

Additive combinatorics. Given two sets $A, B \subseteq \mathbb{F}_2^n$ and an element $x \in \mathbb{F}_2^n$, $A + B \stackrel{\text{def}}{=} \{a+b : a \in A, b \in B\}$ and $x + A \stackrel{\text{def}}{=} \{x+a : a \in A\}$. For any integer t, $tA \stackrel{\text{def}}{=} A + \ldots + A$ where the summation includes A for t times. Studying the structure of tA for small constant t is one of the central topics in additive combinatorics. Readers may refer to the excellent textbook [16]. The following is the famous quasi-polynomial Bogolyubov-Ruzsa lemma due to Sanders [14]. It asserts that $4A$ contains a large subspace if $A \subseteq \mathbb{F}_2^n$ is large. Readers may refer to the nice exposition [9] by Lovett.

Fact 2.9. *[14, 1] Let $A \subseteq \mathbb{F}_2^n$ be a subset of size $|A| = \alpha 2^n$. Then there exists a subspace V of $\mathbb{F}_2^n$ satisfying $V \subseteq 4A$ and*

$$\operatorname{codim}(V) = \mathcal{O}\left(\log^4\left(\alpha^{-1}\right)\right).$$

3 Main Result

Lemma 3.1. *Let $1 \leq c \leq n$, $A_1, A_2, A_3, A_4 \subseteq \mathbb{F}_2^n$ be subsets of size at least 2^{n-c}. Then there exists an affine subspace $V \subseteq A_1 + A_2 + A_3 + A_4$ of $\mathbb{F}_2^n$ such that*

$$\operatorname{codim}(V) = \mathcal{O}\left(c^4\right).$$

Proof. The lemma is trivial if $c \geq n^{1/4}$. We assume that $c < n^{1/4}$. As $|A_1 + A_2| \leq 2^n$, there exists an element $a \in \mathbb{F}_2^n$ such that $a = a_1 + a_2$ for at least 2^{n-2c} pairs $(a_1, a_2) \in A_1 \times A_2$. Then we have $|A_1 \cap (A_2 + a)| \geq 2^{n-2c}$. For the same reason, there exists an element $a' \in \mathbb{F}_2^n$ such that $|A_3 \cap (A_4 + a')| \geq 2^{n-2c}$. Note that $|(A_1 \cap (A_2 + a)) + (A_3 \cap (A_4 + a'))| \leq 2^n$. Thus there exists an element $a'' \in \mathbb{F}_2^n$ such that $a'' = a_3 + a_4$ for at least 2^{n-4c} pairs $(a_3, a_4) \in (A_1 \cap (A_2 + a)) \times (A_3 \cap (A_4 + a'))$. Set

$$A = A_1 \cap (A_2 + a) \cap ((A_3 \cap (A_4 + a')) + a'') = A_1 \cap (A_2 + a) \cap (A_3 + a'') \cap (A_4 + a' + a'').$$

We have $|A| \geq 2^{n-4c} > 0$ since $c < n^{1/4}$. Thus there exists a subspace $V \subseteq 4A$ of codimension $\operatorname{codim}(V) = \mathcal{O}(c^4)$ by Fact 2.9. Note that $4A \subseteq A_1 + A_2 + A_3 + A_4 + a + a'$. The affine subspace $V + a + a'$ serves the purpose. $\square$

We define a downward non-increasing measure $M(\cdot)$ whose 4-th power is an upper bound on $\operatorname{rk}(\cdot)$.

Definition 3.2. Given a function $f : V \to \mathbb{F}_2$, where V is an affine subspace of $\mathbb{F}_2^n$ and $t \stackrel{\text{def}}{=} \dim(V)$, let $L : \mathbb{F}_2^t \to \mathbb{F}_2^n$ be an affine map satisfying $L(\mathbb{F}_2^t) = V$. Set $F_4 : (\mathbb{F}_2^t)^4 \to \mathbb{F}_2$ by $F_4(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} f(L(x_1 + x_2 + x_3 + x_4))$. The complexity of f is defined to be $M(f) \stackrel{\text{def}}{=} \mathrm{CC}^{(4)}(F_4)$.

Note that the affine map is invertible. The complexity $M(f)$ does not depend on the choice of the affine map.

Lemma 3.3. *$M(\cdot)$ is downward non-increasing.*

Proof. Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a boolean function and $V \subseteq \mathbb{F}_2^n$ be an affine subspace. It suffices to show that $M(f) \geq M(f|_V)$. Let F and F' be the 4-argument functions given by Definition 3.2 corresponding to f and $f|_V$, respectively. Assume that $L(z) \stackrel{\text{def}}{=} Az + b$ is the corresponding affine map in Definition 3.2. Given input $(x_1, x_2, x_3, x_4) \in (\mathbb{F}_2^t)^4$, where $t = \dim V$, player P_1 computes $x_1' = A_1 x_1 + b$ and players P_i computes $x_i' = A x_i$ for $i = 2, 3, 4$. Note that $L(x_1 + x_2 + x_3 + x_4) = Ax_1 + Ax_2 + Ax_3 + Ax_4 + b$. We have $F_4'(x_1, x_2, x_3, x_4) = f(x_1' + x_2' + x_3' + x_4') = F_4(x_1', x_2', x_3', x_4')$. The players simulate the protocol that computes F_4 on input (x_1', x_2', x_3', x_4') and get $F_4'(x_1, x_2, x_3, x_4)$. Thus $M(f|_V) = \mathrm{CC}^{(4)}(F_4') \leq \mathrm{CC}^{(4)}(F_4) = M(f)$. $\square$

Lemma 3.4. *For any $f : V \to \mathbb{F}_2$, where V is an affine subspace of $\mathbb{F}_2^n$, it holds that $\mathrm{C}_{\oplus,\min}(f) = \mathcal{O}\left(M(f)^4\right)$. Combining with Fact 2.8, we have $\operatorname{rk}(f) = \mathcal{O}\left(M(f)^4\right)$.*

Proof. We assume w.l.o.g. that $V = \mathbb{F}_2^n$. Let $F_4(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} f(x_1 + x_2 + x_3 + x_4)$. Let $c \stackrel{\text{def}}{=} \mathrm{CC}^{(4)}(F_4)$. The optimal protocol partitions the domain into at most 2^c monochromatic hyperrectangles. Thus there exists a monochromatic hyperrectangle $A_1 \times A_2 \times A_3 \times A_4$ satisfying $|A_1 \times A_2 \times A_3 \times A_4| \geq 2^{4n-c}$. Hence $|A_i| \geq 2^{n-c}$ for $1 \leq i \leq 4$. Using Lemma 3.1, there exists an affine subspace $V \subseteq A_1 + A_2 + A_3 + A_4$ satisfying $\operatorname{codim}(V) = \mathcal{O}(c^4)$. It implies that $\mathrm{C}_{\oplus,\min}(f) = \mathcal{O}(c^4)$. The result follows. ☐

Combining Fact 2.7, Lemma 3.3 and Lemma 3.4, we have

$$\mathrm{D}_{\oplus}(f) \leq \mathcal{O}\left(M(f)^4 \cdot \log \|\hat{f}\|_0\right).$$

By Definition 3.2, $M(f) \leq \mathrm{CC}^{(4)}(F_4)$. Note that $\log \|\hat{f}\|_0 = \log \operatorname{rk}(M_{F_2}) \leq \mathrm{CC}^{(2)}(F_2) \leq \mathrm{CC}^{(4)}(F_4)$ where the equality is by Fact 1.2. The main theorem follows.

Open Problems

Here we list two open problems related to the Log-Rank Conjecture for XOR-functions.

1. The most interesting work along this line is to show that the PDT complexity of f and the communication complexity of the corresponding 2-argument XOR-function F_2 are polynomially equivalent.

2. Can we extend Theorem 1.4 to the randomized communication complexity?

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